

A planar large sieve and sparsity of time-frequency representations

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Abstract – With the aim of measuring the sparsity of a real signal, Donoho and Logan introduced the concept of maximum Nyquist density, and used it to extend Bombieri’s principle of the large sieve to bandlimited functions. This led to several recovery algorithms based on the minimization of the L_1 -norm. In this paper we introduce the concept of planar maximum Nyquist density, which measures the sparsity of the time-frequency distribution of a function. We obtain a planar large sieve principle which applies to time-frequency representations with a gaussian window, or equivalently, to Fock spaces, allowing for perfect recovery of the short-Fourier transform (STFT) of functions in the modulation space M_1 (also known as Feichtinger’s algebra S_0) corrupted by sparse noise and for approximation of missing STFT data in M_1 , by L_1 -minimization.

I. INTRODUCTION

With the aim of measuring the sparsity of a real signal, Donoho and Logan introduced the concept of *maximum Nyquist density*, defined in [12] as

$$\begin{aligned} \rho(T, W) &:= W \cdot \sup_{t \in \mathbb{R}} |T \cap [t, t + 1/W]| \\ &\leq W \cdot |T|, \end{aligned}$$

where $T \subset \mathbb{R}$ and W is the band-size in the space of band-limited functions

$$B_W := \left\{ f \in L^1 : \hat{f}(\omega) = 0, \forall |\omega| > \pi W \right\}.$$

If the set T is sparse in terms of low Lebesgue measure (small concentration in any interval of length $1/W$), then

$\rho(T, W)$ can be considerably small compared to the natural Nyquist density $W \cdot |T|$. We will write $P_A f = \chi_A f$ for the multiplication by the indicator function of A . In [12, Theorem 7], Donoho and Logan proved that, if $f \in B_W$ and $\delta < \frac{2}{W}$, the inequality

$$\|P_T f\|_1 \leq \sup_{t \in \mathbb{R}} |T \cap [t, t + \theta]| \cdot \frac{\pi W/2}{\sin(\pi W \theta/2)} \cdot \|f\|_1 \quad (1)$$

holds. In particular, if $\delta(T)$ denotes the norm of the projection operator P_T and $\theta = 1/W$, then $\delta(T) \leq \frac{\pi}{2} \rho(T, W)$. Note that the inequality (1) falls within the realm of quantitative uncertainty principles [11], [20], [21], [23], [36], which paved the way to the modern theory of compressed sensing (see [9, Section 1.6] or [10], [25]).

Donoho and Logan’s interest in such inequalities, in particular in obtaining good constants depending on the sparsity of the set T , was motivated by signal recovery problems [11]. As an application, they derived the following results, which allows to perfectly reconstruct a bandlimited signal corrupted by sparse noise using L_1 -norm minimization.

Corollary: ([12, Corollary 1]) Suppose that $g = b + n$ is observed, $b \in B_W$, $n \in L_1$, and that the *unknown support* T of the noise n satisfies

$$\rho(T, W) < 1/\pi. \quad (2)$$

Then the solution of the minimization problem

$$\beta(g) = \arg \min_{b \in B_W} \|g - \tilde{b}\|_1$$

is unique and recovers the signal b perfectly ($\beta(g) = b$).

This extends the so-called Logan’s phenomenon ([28], see also the discussion in [11, Section 6.2]). The following

recovery result for missing data extends the results from [11] from the L_2 to the L_1 setting. We will give a proof in the STFT context in Corollary 2.

Corollary: Suppose that we observe $h = P_{T^c}(b + n)$, where $b \in B_W$, $\|n\| \leq \varepsilon$ and

$$\rho(T, W) < 2/\pi.$$

Then, any solution of the minimization problem

$$\sigma(h) := \arg \min_{\tilde{b} \in B_W} \|P_{T^c}(\tilde{b} - h)\|_1$$

satisfies

$$\|b - \sigma(h)\|_1 \leq \frac{4\varepsilon}{2 - \pi \cdot \rho(T, W)}.$$

Let $\varphi(t) = 2^{1/4}e^{-\pi t^2}$ be the normalized gaussian. The *short-time Fourier transform (STFT)* is defined as follows

$$V_\varphi f(x, \omega) = \int_{\mathbb{R}} f(t) \overline{\varphi(t - x)} e^{-2\pi i \omega t} dt. \quad (3)$$

Moreover, define the modulation spaces

$$M^p := \{f \in \mathcal{S}'(\mathbb{R}) : \|V_\varphi f\|_p < \infty\}, \quad p \geq 1.$$

Modulation spaces are ubiquitous in time-frequency analysis [16], [26]. They were introduced in [14]. It is the purpose of this paper to obtain a planar version of (1) and apply it to recovery problems for the short-time Fourier transform of functions in M^1 using L_1 -minimization. The space M^1 is also known as Feichtinger's algebra S_0 and it can be identified with the Bargmann-Fock space $\mathcal{F}_1(\mathbb{C})$ of entire functions.

As a planar analogue of $\rho(T, W)$, we introduce the following concept. Let $\Delta \subset \mathbb{R}^2$. The planar maximum Nyquist density $\rho(\Delta, R)$ is defined as

$$\rho(\Delta, R) := \sup_{z \in \mathbb{R}^2} |\Delta \cap (z + D_{1/R})| \leq |\Delta|, \quad (4)$$

where $D_{1/R} \subset \mathbb{R}^2$ is the disc of radius $1/R$ centered in the origin. If the set Δ is sparse in the sense of Lebesgue measure (small concentration in any disc of radius $1/R$) then $\rho(\Delta, R)$ can be considerably smaller than the natural Nyquist density $|\Delta|$ (see [3], [13], [17], [22], [31], [33] for natural Nyquist densities in the context of Fock and modulation spaces and [18] for a survey on the current state of the art of the topic). Our main result is the following.

Theorem 1: Consider $\Delta \subset \mathbb{R}^2$ and let $f \in M^1$, then, for every $0 < R < \infty$, it holds

$$\|P_\Delta(V_\varphi f)\|_1 \leq \frac{\rho(\Delta, R)}{1 - e^{-\pi/R^2}} \cdot \|V_\varphi f\|_1. \quad (5)$$

Set

$$\delta(\Delta) := \sup_{f \in M^1} \frac{\|P_\Delta(V_\varphi f)\|_1}{\|V_\varphi f\|_1}.$$

By Theorem 1,

$$\delta(\Delta) \leq (1 - e^{-\pi/R^2})^{-1} \rho(\Delta, R). \quad (6)$$

Moreover, if $\delta(\Delta) < \frac{1}{2}$ then every $f \in M^1$ satisfies $\|P_\Delta(V_\varphi f)\|_1 < \|P_{\Delta^c}(V_\varphi f)\|_1$. Combined with the argument in [11, Section 6.2], this implies the following result, which allows to perfectly reconstruct the STFT of a signal in M^1 corrupted by sparse noise, using L_1 -norm minimization.

Corollary 1: Suppose that $G = V_\varphi f + N$ is observed, where $f \in M^1$, $N \in L_1(\mathbb{R}^2)$ and that the *unknown support* Δ of N satisfies

$$\rho(\Delta, R) < \frac{1}{2}(1 - e^{-\pi/R^2}), \quad (7)$$

for some $R > 0$. Then $\delta(\Delta) < \frac{1}{2}$ and the solution of the minimization problem

$$\beta(G) = \arg \min_{g \in M^1} \|G - V_\varphi g\|_1$$

is unique and recovers the signal f perfectly ($\beta(G) = f$).

One can also derive an analogue for the recovery of missing data.

Corollary 2: Let $f \in M^1$ and suppose that one observes $H = P_{\Delta^c}(V_\varphi f + N)$, where $\|N\|_1 \leq \varepsilon$ and that the domain Δ of missing data satisfies

$$\rho(\Delta, R) < (1 - e^{-\pi/R^2}), \quad (8)$$

for some $R > 0$. Then any solution of

$$\sigma(H) = \arg \min_{h \in M^1} \|P_{\Delta^c}(H - V_\varphi h)\|_1$$

satisfies

$$\|V_\varphi(f - \sigma(H))\|_1 \leq \frac{2\varepsilon(1 - e^{-\pi/R^2})}{1 - e^{-\pi/R^2} - \rho(\Delta, R)}.$$

Proof: First, observe that

$$\|P_{\Delta^c}(H - V_\varphi \sigma(H))\|_1 \leq \|n\|_1 \leq \varepsilon.$$

Hence,

$$\|V_\varphi(f - \sigma(H))\|_1$$

$$\begin{aligned}
&= \|P_{\Delta^c} V_\varphi(f - \sigma(H))\|_1 + \|P_\Delta V_\varphi(b - \sigma(g))\|_1 \\
&\leq \|P_{\Delta^c}(V_\varphi f - H)\|_1 + \|P_{\Delta^c}(H - V_\varphi \sigma(H))\|_1 \\
&\quad + \delta(\Delta) \|V_\varphi(f - \sigma(H))\|_1 \\
&\leq 2\varepsilon + \delta(\Delta) \|V_\varphi(f - \sigma(H))\|_1,
\end{aligned}$$

which concludes the proof using (6) and (8).

There are other approaches to the recovery of sparse time-frequency representations which concentrate on the set-up of finite sparse time-frequency representations [29], [30].

Another consequence of Theorem 1 is the following refined L_1 uncertainty principle for the STFT (see [26, Proposition 3.3.1] and [7], [19], [32] for other uncertainty principles for the STFT).

Corollary 3: Suppose that $f \in M^1$ satisfies $\|V_\varphi f\|_1 = 1$ and that $\Delta \subset \mathbb{R}^2$ and $\varepsilon \geq 0$ are such that

$$1 - \varepsilon \leq \int_\Delta |V_\varphi f(x, \omega)| dx d\omega,$$

then

$$1 - \varepsilon \leq \inf_{R>0} \left(\frac{\rho(\Delta, R)}{1 - e^{-\pi/R^2}} \right) \leq |\Delta|.$$

In particular, Corollary 3 shows that the mass of the STFT of a function cannot be concentrated on sets that are locally small over the whole time-frequency plane.

Our arguments to prove Theorem 1 are an adaptation of Selberg's argument for the large sieve (see [6], [8], [24]), along the lines of [12]. The analysis reveals that, at least in the continuous case, dealing with joint time-frequency representations leads to considerable simplifications, due to the existence of local reproducing formulas [33]. This is not surprising since, as observed earlier by Daubechies [13] and Seip [33], the study of joint time-frequency restriction operators with a gaussian window tends to be simplified. In particular, the functions best concentrated in a disc have a simple explicit formula when written in the phase space. This is in contrast with the classical time and band-limiting problem which has been studied in detail by Landau [27] (see also [4] for an alternative approach).

II. MODULATION AND FOCK SPACES

We will follow notations and definitions from [26]. The *Bargmann transform* on \mathbb{C} is defined by

$$\mathcal{B}f(z) = 2^{1/4} \int_{\mathbb{R}} f(t) e^{2\pi t \cdot z - \pi t^2 - \pi z^2/2} dt. \quad (9)$$

Writing $z = x + i\omega$, a simple calculation shows that

$$e^{-i\pi x\omega} V_\varphi f(x, -\omega) = \mathcal{B}f(z) e^{-\pi|z|^2/2}. \quad (10)$$

Let $\mathcal{F}_p(\mathbb{C})$ be the space of entire functions equipped with the norm

$$\|F\|_{\mathcal{F}_p}^p := \int_{\mathbb{C}} |F(z)|^p e^{-\pi p|z|^2/2} dz,$$

if $1 \leq p < \infty$ and

$$\|F\|_{\mathcal{L}_\infty} := \sup_{z \in \mathbb{C}} |F(z)| e^{-\pi|z|^2/2},$$

if $p = \infty$. The Bargmann transform is a unitary operator from $L^2(\mathbb{R})$ to $\mathcal{F}_2(\mathbb{C})$ and extends to a bijective operator from M^p to $\mathcal{F}_p(\mathbb{C})$, for $p \geq 1$ (see [35], or [1] for a proof that extends to polyanalytic Fock spaces). As in [34], we define the translation operator \mathcal{T}_w on $\mathcal{F}_p(\mathbb{C})$ as follows:

$$\mathcal{T}_w F(z) := e^{\pi \bar{w} \cdot z - \pi |w|^2/2} F(z - w).$$

It acts isometrically on every $\mathcal{F}_p(\mathbb{C})$, $p \geq 1$. The corresponding convolution is

$$\begin{aligned}
F * G(z) &:= \int_{\mathbb{C}} F(w) G(z - w) e^{\pi(z \cdot \bar{w} - |w|^2)} dw \\
&= \int_{\mathbb{C}} F(w) \mathcal{T}_w G(z) e^{-\pi|w|^2/2} dw. \quad (11)
\end{aligned}$$

III. PROOF SKETCH OF MAIN RESULTS

A. Concentration estimates

We say that a function $G \in \mathcal{L}_1(\mathbb{C})$ is concentrated on $\Omega \subset \mathbb{C}$ if $\|(I - P_\Omega)G\|_{\mathcal{L}_1} = 0$. Our main results will follow from the following statement which corresponds to (1). We will give a full proof and more general results in [5].

Proposition 1: Suppose that there exists $G \in \mathcal{L}_\infty(\mathbb{C})$ which is concentrated on Ω , such that $F \mapsto F * G$, $\mathcal{F}_1(\mathbb{C}) \rightarrow \mathcal{F}_1(\mathbb{C})$ is bounded and boundedly invertible. Then

$$\int_{\mathbb{C}} |F| d\mu \leq \|G\|_{\mathcal{L}_\infty} \cdot \Lambda(\mu, \Omega) \cdot \nu(G) \cdot \|F\|_{\mathcal{L}_1},$$

where $\nu(G) := \sup_{\Phi \in \mathcal{F}_1(\mathbb{C})} \left(\frac{\|\Phi\|_{\mathcal{L}_1}}{\|\Phi * G\|_{\mathcal{L}_1}} \right)$ and

$$\Lambda(\mu, \Omega) := \sup_{w \in \mathbb{C}} \left(\int_{w+\Omega} e^{\pi|z|^2/2} d\mu(z) \right).$$

Proof sketch: For $F \in \mathcal{F}_1(\mathbb{C})$, there exists $F^* \in \mathcal{F}_1(\mathbb{C})$ unique such that $F = F^* * G$. Hence, replacing F by $F^* * G$ and using, one after another, $\|(I - P_\Omega)G\|_{\mathcal{L}_1} = 0$, Fubini's theorem, and Hölder's inequality ($p = 1$) yields

$$\int_{\mathbb{C}} |F(z)| d\mu(z) \leq \|G\|_{\mathcal{L}_\infty} \cdot \Lambda(\mu, \Omega) \cdot \|F^*\|_{\mathcal{L}_1}.$$

The observation that $\frac{\|F\|_{\mathcal{L}_1}}{\|F^* * G\|_{\mathcal{L}_1}} = 1$ thus implies our statement.

B. Proof of Theorem 1

Define

$$d\mu(z) := \chi_\Delta(z) e^{-\pi|z|^2/2} dz,$$

with $\Delta \subset \mathbb{C}$ some subset of nonzero measure. Consequently, $\|F\|_{\mathcal{L}_{1,\mu}} = \|P_\Delta F\|_{\mathcal{L}_1}$ and setting $\Omega = D_{1/R}$ yields

$$\Lambda(\mu, D_{1/R}) = \sup_{z \in \mathbb{C}} |\Delta \cap (z + D_{1/R})| = \rho(\Delta, R).$$

Let F be entire and $R > 0$, then for any $z \in \mathbb{C}$, the following local reproducing formula holds [34]:

$$F(z) = (1 - e^{-\pi/R^2})^{-1} \cdot (F * \chi_{D_{1/R}})(z). \quad (12)$$

Now, let $R > 0$, choosing $G = G_R := \chi_{D_{1/R}}$ yields that convolution with G_R gives a bounded and invertible operator on $\mathcal{F}_1(\mathbb{C})$. Then $\|G_R\|_{\mathcal{L}_\infty} = 1$,

$$\nu(G_R) = \sup_{\Phi \in \mathcal{F}_1} \left(\frac{\|\Phi\|_{\mathcal{L}_1}}{\|\Phi * G_R\|_{\mathcal{L}_1}} \right) = \frac{1}{1 - e^{-\pi/R^2}},$$

and Proposition 1 yields

$$\frac{\|P_\Delta F\|_{\mathcal{L}_1}}{\|F\|_{\mathcal{L}_1}} \leq \frac{\rho(\Delta, R)}{1 - e^{-\pi/R^2}}.$$

This proves the result for $F \in \mathcal{F}_1(\mathbb{C})$. Since the Bargmann transform extends to a bijective operator from M^1 to $\mathcal{F}_1(\mathbb{C})$, there exists $f \in M^1$ such that

$$F(z) = \mathcal{B}f(z) = e^{-i\pi x\omega + \pi|z|^2/2} V_\varphi f(x, -\omega).$$

This completes the proof.

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REFERENCES

- [1] L. D. Abreu, K. Gröchenig. Banach Gabor frames with Hermite functions: polyanalytic spaces from the Heisenberg group. *Appl. Anal.*, 91: 1981-1997, 2012.
- [2] L. D. Abreu, M. Dörfler. An inverse problem for localization operators. *Inverse Problems*, 28: 115001, 2012.
- [3] L. D. Abreu, K. Gröchenig, J. L. Romero. On accumulated spectrograms. *Trans. Amer. Math. Soc.*, 368: 3629-3649, 2016.
- [4] L. D. Abreu, J. M. Pereira. Measures of localization and quantitative Nyquist densities. *Appl. Comput. Harmon. Anal.*, 38 (3): 524-534, 2015.
- [5] L. D. Abreu, M. Speckbacher. *In preparation*.
- [6] C. Aubel, H. Bölcskei. Vandermonde matrices with nodes in the unit disk and the large sieve. *arXiv:1701.02538 preprint*, 2017.
- [7] A. Bonami, B. Demange, P. Jaming. Hermite functions and uncertainty principles for the Fourier and the windowed Fourier transforms. *Rev. Math. Iberoamericana*, 19: 23-55, 2003.
- [8] E. Bombieri. *Le grand crible dans la théorie analytique des nombres*. Société Mathématique de France, 1974.
- [9] E. J. Candès, J. Romberg, T. Tao. Robust uncertainty principles: Exact signal reconstruction from highly incomplete frequency information. *IEEE Trans. Inf. Theor.*, 52: 489-509, 2006.
- [10] D. L. Donoho. Compressed Sensing. *IEEE Trans. Inform. Theor.*, 52(4): 1289-1306, 2006.
- [11] D. L. Donoho, P. B. Stark. Uncertainty principles and signal recovery. *SIAM J. Appl. Math.*, 49: 906-931, 1989.
- [12] D. L. Donoho, B. F. Logan. Signal recovery and the large sieve. *SIAM J. Appl. Math.*, 52: 577-591, 1992.
- [13] I. Daubechies. Time-frequency localization operators: a geometric phase space approach. *IEEE Trans. Inform. Theor.*, 34: 605-612, 1988.
- [14] H. G. Feichtinger. Modulation spaces on locally compact abelian groups. In *Proceedings of "International Conference on Wavelets and Applications" 2002*, pages 99-140, Chennai, India, 2003. Updated version of a technical report, University of Vienna, 1983.
- [15] H. G. Feichtinger. On a new Segal algebra. *Monatsh. Math.* 92(4): 269-289, 1981.
- [16] H. G. Feichtinger, K. Gröchenig. Banach spaces related to integrable group representations and their atomic decompositions, I. *J. Funct. Anal.*, 86(2): 307-340, 1989.
- [17] F. DeMari, H. G. Feichtinger, K. Nowak. Uniform eigenvalue estimates for time-frequency localization operators, *J. London Math. Soc.*, 65: 720-732, 2002.
- [18] H. Führ, K. Gröchenig, A. Haimi, A. Klotz, J. L. Romero. Density of sampling and interpolation in reproducing kernel Hilbert spaces. *arXiv:1607.07803v2 preprint*, 2016.
- [19] K. Gröchenig, E. Malinnikova. Phase space localization of Riesz basis for $L^2(\mathbb{R}^d)$. *Rev. Mat. Iberoam.*, 29: 1003-1031, 2013.
- [20] P. Jaming, A. Powell. Time-frequency concentration of generating systems. *Proc. Amer. Math. Soc.*, 139: 3279-3990, 2011.
- [21] P. Jaming, A. Powell. Uncertainty principles for orthonormal sequences. *J. Funct. Anal.*, 243: 611-630, 2007.
- [22] Y. Lyubarskii. Frames in the Bargmann space of entire functions. *Entire and subharmonic functions*, 167-180, Adv. Soviet Math., 11, Amer. Math. Soc., Providence, RI (1992).
- [23] E. Malinnikova. Orthonormal sequences in $L^2(\mathbb{R}^d)$ and time frequency localization. *J. Fourier Anal. Appl.*, 16: 983-1006, 2010.

- [24] H. L. Montgomery. The analytic principle of the large sieve. *Bull. Amer. Math. Soc.*, 84: 547-567, 1978.
- [25] S. Foucart, H. Rauhut. *A Mathematical Introduction to Compressed Sensing*. Springer, New York, 2013.
- [26] K. Gröchenig, *Foundations of Time-Frequency Analysis*. Birkhäuser, Boston, 2001.
- [27] H. J. Landau. Necessary Density Conditions for Sampling and Interpolation of Certain Entire Functions. *Acta Math.*, 117: 37-52, 1967.
- [28] B. F. Logan. *Properties of high-pass signals*. Ph.D. thesis, Columbia University, New York, 1965.
- [29] G. E. Pfander, H. Rauhut. Sparsity in time-frequency representations. *J. Four. Anal. Appl.*, 16: 233-260, 2010.
- [30] G. E. Pfander, H. Rauhut, J. Tanner. Identification of matrices having a sparse representation. *IEEE Trans. Signal Proc.*, 56: 5376-5388, 2008.
- [31] J. Ramanathan, T. Steger. Incompleteness of sparse coherent states. *Appl. Comp. Harm. Anal.*, 2: 148-153, 1995.
- [32] B. Ricaud, B. Torresani. A survey of uncertainty principles and some signal processing applications. *Adv. Comp. Math.*, 40(3): 629-650, 2014.
- [33] K. Seip. Reproducing formulas and double orthogonality in Bargmann and Bergman spaces. *SIAM J. Math. Anal.*, 22: 856-876, 1991.
- [34] K. Seip, R. Wallstén. Density Theorems for sampling and interpolation in the Bargmann-Fock space I, *J. Reine Angew. Math.*, 429: 91-106, 1992.
- [35] M. Signahl, J. Toft. Mapping properties for the Bargmann transform on modulation spaces. *J. Pseudo Diff. Oper. Appl.*, 3: 1-30, 2012.
- [36] T. Tao. An uncertainty principle for cyclic groups of prime order. *Math. Res. Lett.*, 12: 121-127, 2005.